Lecture 17

Implicit differentiation.

Making y the subject: If $xy = 1,\ y = x^{-1}$ & $\frac{dy}{dx} = -x^{-2}$. But $xy - y^2 = 1$ is harder to be changed to the subject of y.

Note: $\frac{d}{dx}(f(y)) = f'(y) \cdot \frac{dy}{dx}$

Example 1. Find $\frac{dy}{dx}$ given $xy - y^2 = 1$.

$$\begin{aligned} 1 \cdot y + x \cdot 1 \cdot \frac{dy}{dx} - 2y \cdot \frac{dy}{dx} &= 0 \\ x\frac{dy}{dx} - 2y\frac{dy}{dx} &= -y \\ \frac{dy}{dx} &= \frac{-y}{x-2y} \\ &= \frac{y}{2y-x} \quad \square \end{aligned}$$

Example 2. Find $\frac{dy}{dx}$ if $x^2y^3 = 1$.

$$\begin{aligned} 2xy^3 + x^2 \cdot 3y^2\frac{dy}{dx} &= 0 \\ 3x^2y^2\frac{dy}{dx} &= -2xy^3 \\ \frac{dy}{dx} &= \frac{-2xy^3}{3x^2y^2} \\ &= \frac{-2y}{3x} \quad \square \end{aligned}$$

Lecture 18

Implicit functions examples from Set 6M (Coroneos, 1982c)

Question 7. If $x^2y = (x+y)^3$, show that $\frac{dy}{dx} = \frac{y}{x}$.

$$\begin{aligned}
\tfrac{d}{dx}x^2y &= \tfrac{d}{dx}(x+y)^3. \\
\therefore\ y(\tfrac{d}{dx}x^2) + x^2\tfrac{d}{dy}y\tfrac{dy}{dx} &= \tfrac{d}{dx}(x^3+3x^2y+3xy^2+y^3). \\
\therefore\ 2xy + x^2\tfrac{dy}{dx} &= \tfrac{d}{dx}x^3 + \tfrac{d}{dx}3x^2y + \tfrac{d}{dx}3xy^2 + \tfrac{d}{dx}y^3 \\
&= 3x^2 + y\tfrac{d}{dx}3x^2 + 3x^2\tfrac{d}{dy}y\tfrac{dy}{dx} + y^2\tfrac{d}{dx}3x + 3x\tfrac{d}{dy}y^2\tfrac{dy}{dx} + \tfrac{d}{dy}y^3\tfrac{dy}{dx} \\
&= 3x^2 + 6xy + 3x^2\tfrac{dy}{dx} + 3y^2 + 6xy\tfrac{dy}{dx} + 3y^2\tfrac{dy}{dx}. \\
\therefore\ \tfrac{dy}{dx}(2x^2+6xy+3y^2) &= -3x^2-4xy-3y^2. \\
\therefore\ \tfrac{dy}{dx} &= \tfrac{-3x^2-4xy-3y^2}{2x^2+6xy+3y^2} \\
&= \tfrac{-3(x+y)^2+2xy}{3(x+y)^2-x^2}\cdot\tfrac{x+y}{x+y} \\
&= \tfrac{-3(x+y)^3+2x^2y+2xy^2}{3(x+y)^3-x^3-x^2y} \\
&= \tfrac{-3x^2y+2x^2y+2xy^2}{3x^2y-x^3-x^2y} \quad (\text{since } x^2y=(x+y)^3) \\
&= \tfrac{2xy^2-x^2y}{2x^2y-x^3} \\
&= \tfrac{2y^2-xy}{2xy-x^2} \\
&= \tfrac{y}{x}\cdot\tfrac{2y-x}{2y-x} \\
&= \tfrac{y}{x}. \quad \square
\end{aligned}$$

Question 8. If $xy + 5x + 8 = 0$, show that $x^2\frac{dy}{dx} = 8$.

$$\begin{aligned}
\tfrac{d}{dx}(xy+5x+8) &= \tfrac{d}{dx}0. \\
\therefore\ y\tfrac{d}{dx}x + x\tfrac{d}{dy}y\tfrac{dy}{dx} + \tfrac{d}{dx}5x + \tfrac{d}{dx}8 &= 0. \\
\therefore\ y + x\tfrac{dy}{dx} + 5 &= 0. \\
\therefore\ x\tfrac{dy}{dx} &= -y-5. \\
\therefore\ \tfrac{dy}{dx} &= \tfrac{-y-5}{x} \\
\therefore\ x^2\tfrac{dy}{dx} &= x^2\left(\tfrac{-y-5}{x}\right) \\
&= -xy-5x \\
&= 8 \quad (\text{since } xy+5x+8=0). \quad \square
\end{aligned}$$

Question 9. If $ax^2 + by^2 = c$ show that $\frac{d^2y}{dx^2} = \frac{-ac}{b^2y^3}$.

$$\begin{aligned}
\tfrac{d}{dx}ax^2 + b\tfrac{d}{dy}y^2\tfrac{dy}{dx} &= \tfrac{d}{dx}c.\\
\therefore 2ax + 2by\tfrac{dy}{dx} &= 0\\
\therefore \tfrac{dy}{dx} &= \tfrac{-2ax}{2by}\\
&= \tfrac{-ax}{by}\\
\therefore \tfrac{d^2y}{dx^2} &= \tfrac{d}{dx}\left(\tfrac{dy}{dx}\right)\\
&= \tfrac{d}{dx}\tfrac{-a}{b}xy^{-1}\\
&= \tfrac{-a}{b}(y^{-1}\cdot 1 + x\cdot \tfrac{dy^{-1}}{dy}\cdot\tfrac{dy}{dx})\\
&= \tfrac{-a}{b}\left(\tfrac{1}{y} - \tfrac{x}{y^2}\cdot\tfrac{-ax}{by}\right)\\
&= -a\left(\tfrac{by^2+ax^2}{b^2y^3}\right)\\
&= \tfrac{-ac}{b^2y^3}\quad (\text{since } ax^2+by^2=c). \quad \square
\end{aligned}$$

Lecture 19
The Conic Sections

The Circle (Cartesian equation)

$$x^2 + y^2 = a^2$$

centre at origin and radius a.

$$(x-h)^2 + (y-k)^2 = a^2$$

centre $(h.k)$ and radius a.

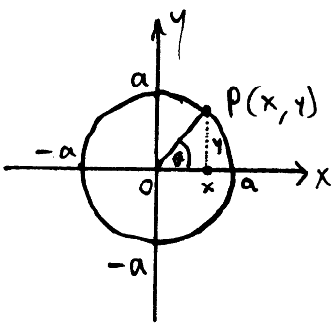
Parametric Equations

Idea of parametric equations is to express x and y in terms of a 3rd variable such as t or θ.

E.g., find the cartesian equation of the following curve:

$$\begin{matrix}\text{parametric} \\ \text{equations}\end{matrix}\begin{cases} x = 1 + t & \Rightarrow t = x - 1 \\ y = t^2 \end{cases}$$

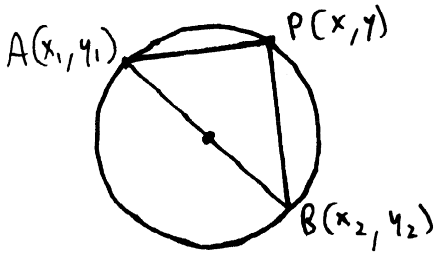
$$\therefore y = (x-1)^2 \text{ - cartesian equation.}$$



$x = a\cos\theta$
$y = a\sin\theta$
-from the diagram.
-parametric equations of a circle centre $(0, 0)$, radius a.

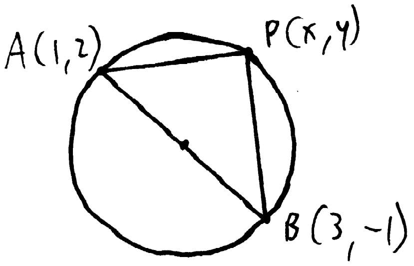
With centre (h, k), parametric equations are:
$x = h + a\cos\theta$
$y = k + a\sin\theta$.

Circle on diameter $A(x_1, y_1), B(x_2, y_2)$:



Since $AP \perp BP$, gradAP.grad$BP = -1$.
$\therefore \frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1$
$(y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$
$\therefore (y-y_1)(y-y_2) + (x-x_1)(x-x_2) = 0$.

Eg. 1. Find the equation of the circle with diameter A, B where $A = (1, 2)$ and $B = (3, -1)$. Find the centre and radius.



$\frac{y-2}{x-1} \cdot \frac{y+1}{x-3} = -1$
$(y-2)(y+1) + (x-1)(x-3) = 0$
$\therefore y^2 - y^2 + x^2 - 4x + 1 = 0$
$\therefore y^2 - y + \frac{1}{2} + x^2 - 4x + 4 + 1 = 4\frac{1}{2}$.
$(x-2)^2 + (y - \frac{1}{2})^2 = 3\frac{1}{4}$ $\therefore$ the centre$= (2, \frac{1}{2})$, radius$= \frac{\sqrt{13}}{2}$. $\square$

Example 1 from Coroneos, 1982b, p25.
Find the equation of the circle
(i) centre the origin to pass through the point $(-3, 7)$
(ii) centre $(1, -4)$ to touch the line $2x - 3y = 9$
(iii) which touches the coordinates axes and passes through $(8, 1)$.

(i) $x^2 + y^2 = (-3-0)^2 + (7-0)^2$ i.e., $x^2 + y^2 = 58$.

(ii) $(x-1)^2+(y+4)^2=\left(\frac{|2(1)-3(-4)-9|}{\sqrt{2^2+(-3)^2}}\right)^2$ i.e., $(x-1)^2+(y+4)^2=\frac{25}{13}$.

(iii) $(x-a)^2+(y-a)^2=a^2$ where $(8-a)^2+(1-a)^2=a^2$.

Hence $64-16a+a^2+1-2a+a^2=a^2$

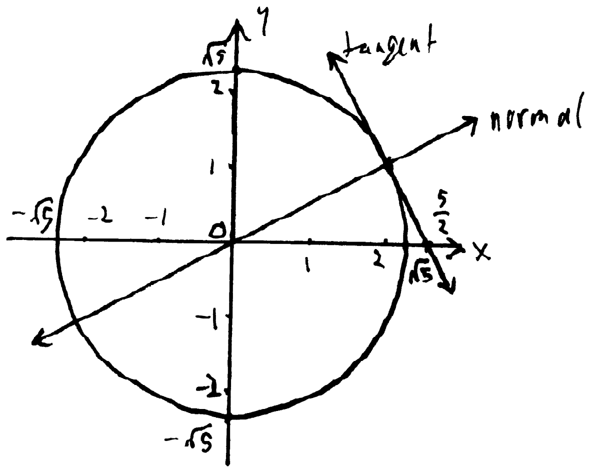
$\therefore a^2-18a+65=0$

$\therefore (a-5)(a-13)=0 \therefore a=5,13$

$\therefore (x-5)^2+(y-5)^2=25$ or $(x-13)^2+(y-13)^2=169$. $\square$

Lecture 20

Eg. 1. Find the equation of the tangent and normal to the circle $x^2 + y^2 = 5$ at $P(2, 1)$.



Since it is a circle with centre $(0, 0)$, then the normal does as well $\therefore$ the equation of the normal can be found and also the tangent since the normal $\perp$ tangent. However by calculus methods,

$x^2 + y^2 = 5$
$\therefore 2x + 2y\frac{dy}{dx} = 0$
$\therefore \frac{dy}{dx} = -\frac{x}{y}$
$\therefore m_T = \frac{-2}{1} = -2$ and $m_N = \frac{1}{2}$.
Using $y - y_1 = m(x - x_1)$:

Tangent

$y - 1 = -2(x - 2)$
$y - 1 = -2x + 4$
$2x + y - 5 = 0$

Normal

$y - 1 = \frac{1}{2}(x - 2)$
$2y - 2 = x - 2$
$x - 2y = 0.$ $\square$

Eg. 2. Find the equation of the tangent and normal to $x^2 + y^2 = a^2$ at
(i) $P(x, y)$ **(ii)** $P(a\cos\theta, a\sin\theta)$.

(i) $x^2 + y^2 = a^2$
$\therefore 2x + 2y\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-x}{y}$
$\therefore m_T = \frac{-x_1}{y_1}$
and $m_N = \frac{y_1}{x_1}$ $\quad \therefore$ since $y - y_1 = m(x - x_1)$

Tangent

$y - y_1 = \frac{-x_1}{y_1}(x - x_1)$
$y_1 y - y_1^2 = -x_1 x + x_1^2$
$x_1 x + y_1 y = x_1^2 + y_1^2 = a^2$
$\therefore x_1 x + y_1 y = a^2$

Normal

$y - y_1 = \frac{y_1}{x_1}(x - x_1)$
$x_1 y - x_1 y_1 = y_1 x - x_1 y_1$
$\therefore y_1 x - x_1 y = 0 \quad \square$

(ii) $\frac{dy}{dx} = \frac{-x}{y}$.
$m_T = \frac{-a\cos\theta}{a\sin\theta} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$
$\therefore m_N = \frac{\sin\theta}{\cos\theta} = \tan\theta$.

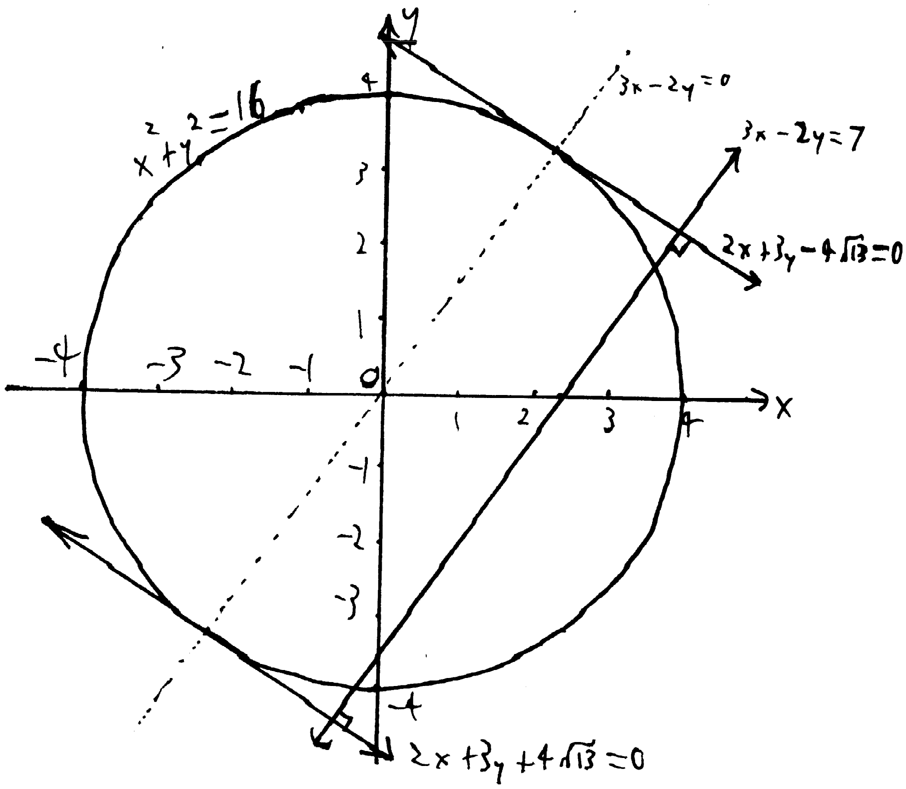
Tangent

$y - y_1 = m(x - x_1)$
$y - a\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - a\cos\theta)$
$y\sin\theta - a\sin^2\theta = -x\cos\theta + a\cos\theta$
$x\cos\theta + y\sin\theta = a\sin^2\theta + a\cos^2\theta$
$\therefore x\cos\theta + y\sin\theta = a$

Normal

$y - y_1 = m(x - x_1)$
$y - a\sin\theta = \frac{\sin\theta}{\cos\theta}(x - a\cos\theta)$
$y\cos\theta - a\sin\theta\cos\theta = x\sin\theta - a\sin\theta\cos\theta$
$\therefore x\sin\theta - y\cos\theta = 0 \quad \square$

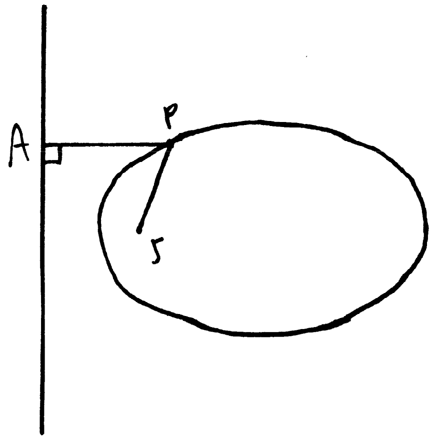
Eg. 3. Find the equation of the tangent to the circle $x^2 + y^2 = 16$ and perpendicular to $3x - 2y = 7$.



Line has $m = \frac{3}{2}$.
$\therefore m_T = \frac{-2}{3}$
$\therefore 2x + 3y = k$
$\therefore 2x + 3y - k = 0$
Perpendicular distance of tangent to centre of circle = radius of circle.
$\therefore d = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}} = 4$
$\therefore \frac{|k|}{\sqrt{13}} = 4$ (since c is constant and x_1 and $y_1 = 0$.)
$k = \pm 4\sqrt{13} \therefore 2x + 3y \pm 4\sqrt{13}$. □

Lecture 21

Ratio of distance from a fixed point (focus) to the distance from a fixed line (directrix) equals a constant (eccentricity, e).

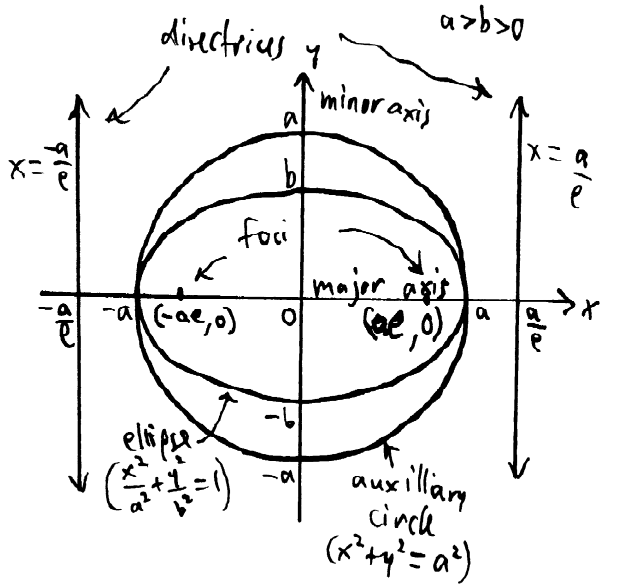


$\frac{PS}{PM} = e$

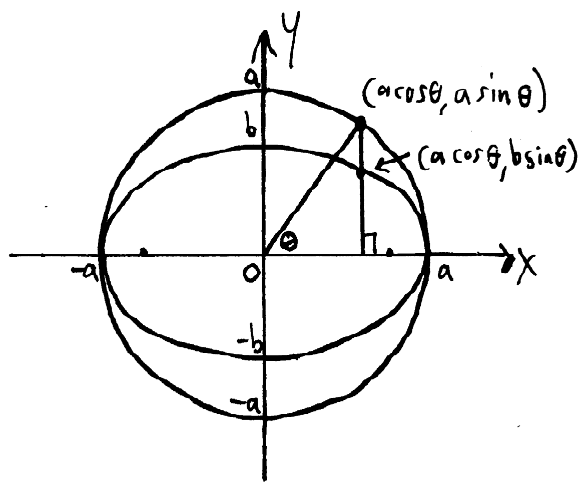
$e > 1 \Rightarrow$ hyperbola
$e = 1 \Rightarrow$ circle
$e < 1 \Rightarrow$ ellipse.

Ellipse.

General equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\qquad a > b.$
where foci $(\pm ae, 0)$
directrix $x = \pm \frac{a}{e}$
eccentricity: $e^2 = 1 - \frac{b^2}{a^2}$
$\therefore e = \sqrt{1 - \frac{b^2}{a^2}}$

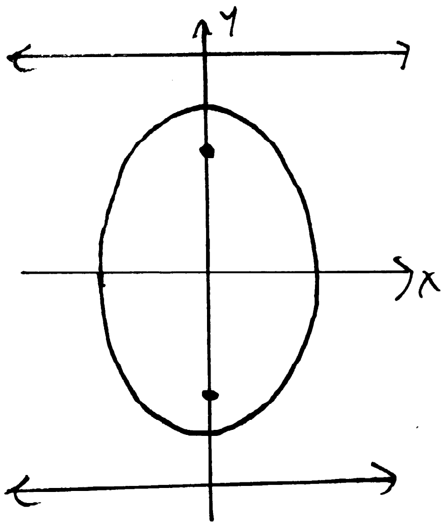


Parametric equation



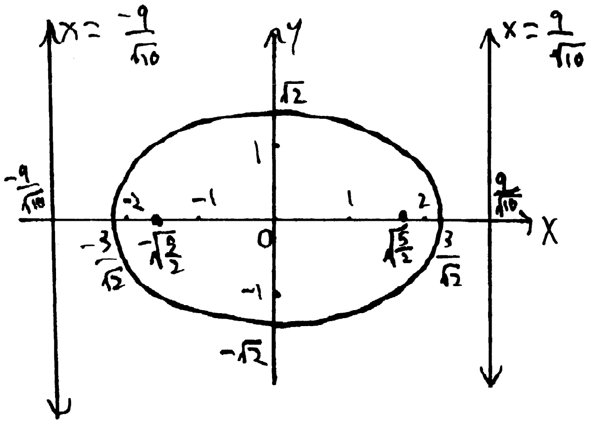
Parametric equations of ellipse are $x = a\cos\theta, y = b\sin\theta$

Note for $b > a$ we have



Eg. Find the eccentricity, foci, directrices, and sketch the ellipse $4x^2 + 9y^2 = 18$. Also find the parametric equation.

$\frac{4x^2}{18} + \frac{9y^2}{18} = 1$
$\frac{2x^2}{9} + \frac{y^2}{2} = 1$
$\frac{x^2}{9/2} + \frac{y^2}{2} = 1$
$\therefore a = \frac{3}{\sqrt{2}}$ and $b = \sqrt{2}$
$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{2}{9/2}} = \frac{\sqrt{5}}{3}$.
foci: $(\pm ae, 0) = \Big(\pm \frac{3}{\sqrt{2}} \cdot \frac{\sqrt{5}}{3}, 0\Big) = \Big(\pm \sqrt{\frac{5}{2}}, 0\Big)$
directrices $x = \pm\frac{a}{e} = \pm\frac{3/\sqrt{2}}{\sqrt{5}/3} = \pm\frac{3}{\sqrt{2}} \cdot \frac{3}{\sqrt{5}} = \pm\frac{9}{\sqrt{10}}$



Parametric equations: $x = \frac{3}{\sqrt{2}}\cos\theta$, $y = \sqrt{2}\sin\theta$, $0 \le \theta < 2\pi$.

Lecture 22

Tangents and Normals to ellipses

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$
$\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}.$

At $P(a\cos\theta, b\sin\theta)$

Tangent:-

$y - y_1 = m(x - x_1)$
$m_{\tan} = \frac{-b^2(\cos\theta}{a^2(b\sin\theta)} = \frac{-b\cos\theta}{a\sin\theta}$
$\therefore$ equation is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta).$
$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$
$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$
$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Normal:-

$m_{\text{norm}} = \frac{-1}{m_{\tan}} = \frac{a\sin\theta}{b\cos\theta}$
$\therefore$ equation is $y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$
$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$
$ax\sin\theta - by\cos\theta - a^2\sin\theta\cos\theta + b^2\sin\theta\cos\theta = 0$
$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$

Tangent:-

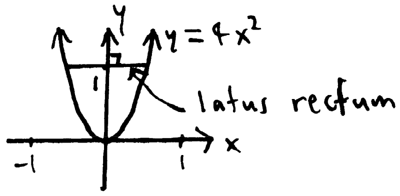
$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$
At $P(x_1, y_1)$, equation is
$y - y_1 = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$
$a^2 yy_1 - a^2 y_1^2 = -b^2 x_1(x - x_1)$
$a^2 yy_1 - a^2 y_1^2 = -b^2 xx_1 + b^2 x_1^2$
$b^2 xx_1 - b^2 x_1^2 + a^2 yy_1 - a^2 y_1^2 = 0$
$b^2 xx_1 + a^2 yy_1 = b^2 x_1^2 + a^2 y_1^2$
$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$ since $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
$\therefore \frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$

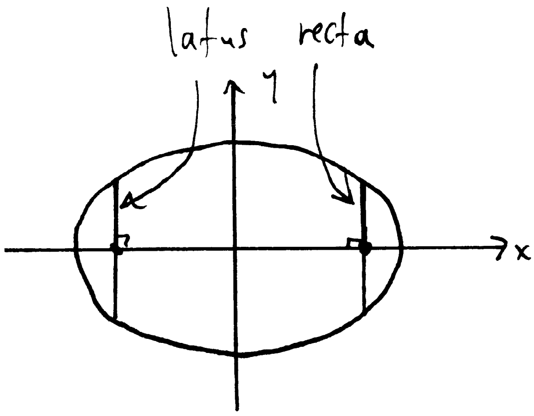
Normal:-

On $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, P(x_1, y_1)$
$m_{\text{norm}} = \frac{-1}{m_{\tan}} = \frac{a^2 y_1}{b^2 x_1}$

and $\therefore (y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$
$b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 y_1 x_1$
$b^2 x_1 y - a^2 y_1 x = b^2 x_1 y_1 - a^2 y_1 x = x_1 y_1 (b^2 - a^2)$
$\frac{b^2 x_1 y - a^2 y_1 x}{x_1 y_1} = b^2 - a^2$
$\therefore \frac{a^2 y_1 x - b^2 x_1 y}{x_1 y_1} = a^2 - b^2$
$\therefore \frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

Note: A latus rectum (plural - latus recta) is the perpendicular chord to the major axis of summetry of the shape passing through the focus (plural - foci).



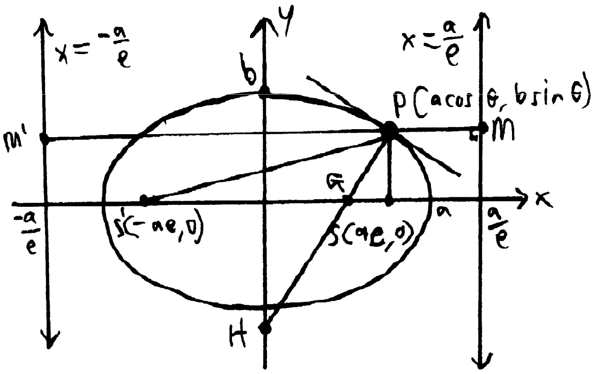


Lecture 23

Example. P is the point $(a\cos\theta, b\sin\theta)$ on $E: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with eccentricity e, foci S, S'.

i. Show that $SP = a(1-e\cos\theta)$ and find an expression for SP. Show that $SP+S'P$ is independent of the position of P.

ii. If the normal at P intersects the major axis at G and minor axis at H, show that $\frac{PG}{PH} = 1-e^2$.



$\frac{PS}{PM} = e$

$\therefore PS = PM.e = (\frac{a}{e} - a\cos\theta)e = a - ae\cos\theta = a(1-e\cos\theta)$

$\frac{PS'}{PM'} = e$

$S'P = ePM' = e(\frac{a}{e} + a\cos\theta) = a + ae\cos\theta = a(1+e\cos\theta)$

$\therefore SP + S'P = a(1-e\cos\theta) + a(1+e\cos\theta) = a(1-e\cos\theta+1+e\cos\theta) = 2a$

which is independant of the position of P.

Normal: $ax\sec\theta - by\text{cosec}\,\theta = a^2-b^2$

major axis: $y=0$

$\therefore ax\sec\theta = a^2-b^2$

$x = \frac{a^2-b^2}{a\sec\theta}$

$\therefore G(\frac{a^2-b^2}{a\sec\theta}, 0)$

Minor axis: $x = 0$ by cosec $\theta = a^2-b^2$

$y = \frac{-(a^2-b^2)}{b\text{cosec}\,\theta}$ and therefore $H = (0, \frac{-(a^2-b^2)}{b\text{cosec}\,\theta})$

$$
\begin{aligned}
\therefore \frac{PG}{PH} &= \sqrt{\frac{(a\cos\theta - \frac{a^2-b^2}{a\sec\theta})^2 + (b\sin\theta)^2}{(a\cos\theta)^2 + (b\sin\theta - (\frac{b^2-a^2}{\operatorname{cosec}\theta}))^2}} \\
&= \sqrt{\frac{(a\cos\theta - a\cos\theta + \frac{b^2}{a}\cos\theta)^2 + (b\sin\theta)^2}{(a\cos\theta)^2 + (b\sin\theta - b\sin\theta + \frac{a^2}{b}\sin\theta)^2}} \\
&= \sqrt{\frac{\frac{b^4}{a^2}\cos^2\theta + b^2\sin^2\theta}{a^2\cos^2\theta + \frac{a^4}{b^2}\sin^2\theta}} \\
&= \sqrt{\frac{b^6\cos^2\theta + a^2b^4\sin^2\theta}{a^4b^2\cos^2\theta + a^6\sin^2\theta}} \\
&= \frac{b^2}{a^2}\sqrt{\frac{b^2\cos^2\theta + a^2\sin^2\theta}{b^2\cos^2\theta + a^2\sin^2\theta}} \\
&= \frac{b^2}{a^2} \\
&= 1 - e^2 \quad \text{since} \;\; e^2 = 1 - \frac{b^2}{a^2}.
\end{aligned}
$$

Lecture 24

From Coroneos, 1982b, Set 2D Q8.

8. P is the point $(a\cos\theta, b\sin\theta)$ on the ellipse E and Q is the corresponding point on the auxiliary circle C (i.e., P, Q have the same abscissa). State the coordinates of Q.

(i) Find the equation of the tangent at P to E and at Q to C. Prove that these tangents meet on the major axis of the ellipse.
(ii) Show that the perpendicular distance from a focus S of E to the tangent at Q to C is equal to SP.
(iii) Find the equation of OQ and of the normal to E at P. Show that these meet on the circle $x^2 + y^2 = (a + b)^2$.

$Q = (a\cos\theta, a\sin\theta)$

(i) When E is the locus of P, the equation of E is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and therefore $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$.
Hence $\frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$ and thus at P, $\frac{dy}{dx} = \frac{-b^2\cos\theta}{a^2 b\sin\theta} = \frac{-b\cos\theta}{a\sin\theta}$.
$\therefore \tan_P : y - b\sin\theta = \frac{-b\sin\theta}{a\cos\theta}(x - a\cos\theta)$. Hence $ay\sin\theta - ab\sin^2\theta = -b\cos\theta + ab\cos^2\theta$.
$\therefore bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta) = ab$.
When C is the locus of Q, the equation of C is $x^2 + y^2 = a^2$ and therefore $2x + 2y\frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{-x}{y}$.
$\therefore$ at Q, $\frac{dy}{dx} = \frac{-a\cos\theta}{a\sin\theta} = \frac{-\cos\theta}{\sin\theta}$
$\therefore \tan_Q :\ y - a\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - a\cos\theta)$
$\therefore\ y\sin\theta - a\sin^2\theta = -x\cos\theta + a\cos^2\theta$
$\therefore x\cos\theta + y\sin\theta = a(\sin^2\theta + \cos^2\theta) = a$.
When $y = 0$ for $\tan_P, bx\cos\theta = ab$
$\therefore x = \frac{a}{\cos\theta}$
and for $\tan_Q$, $x\cos\theta = a$ and therefore $x = \frac{a}{\cos\theta}$ which is the same point for $\tan_P((\frac{a}{\cos\theta}, 0))$ and therefore the two tangents meet on the major axis of the ellipse (x-axis $(y = 0)$).

(ii) When a focus of E is S, S is $(ae, 0)$

$$
\begin{aligned}
\therefore SP &= \sqrt{(a\cos\theta - ae)^2 + (b\sin\theta - 0)^2} \\
&= \sqrt{a^2(\cos\theta - e)^2 + b^2\sin^2\theta} \\
&= \sqrt{a^2(\cos\theta - e)^2 + a^2(1 - e^2)(1 - \cos^2\theta)} \\
&= a\sqrt{\cos^2\theta - 2e\cos\theta + e^2 + 1 - \cos^2\theta - e^2 + e^2\cos^2\theta} \\
&= a\sqrt{e^2\cos^2\theta - 2e\cos\theta + 1} \\
&= a\sqrt{(e\cos\theta - 1)^2} \\
&= a|e\cos\theta - 1|.
\end{aligned}
$$

The perpendicular distance from S to $\tan_Q$ is $\left|\frac{ae\cos\theta + 0\sin\theta - a}{\sqrt{\sin^2\theta + \cos^2\theta}}\right| = a|e\cos\theta - 1| = SP$.

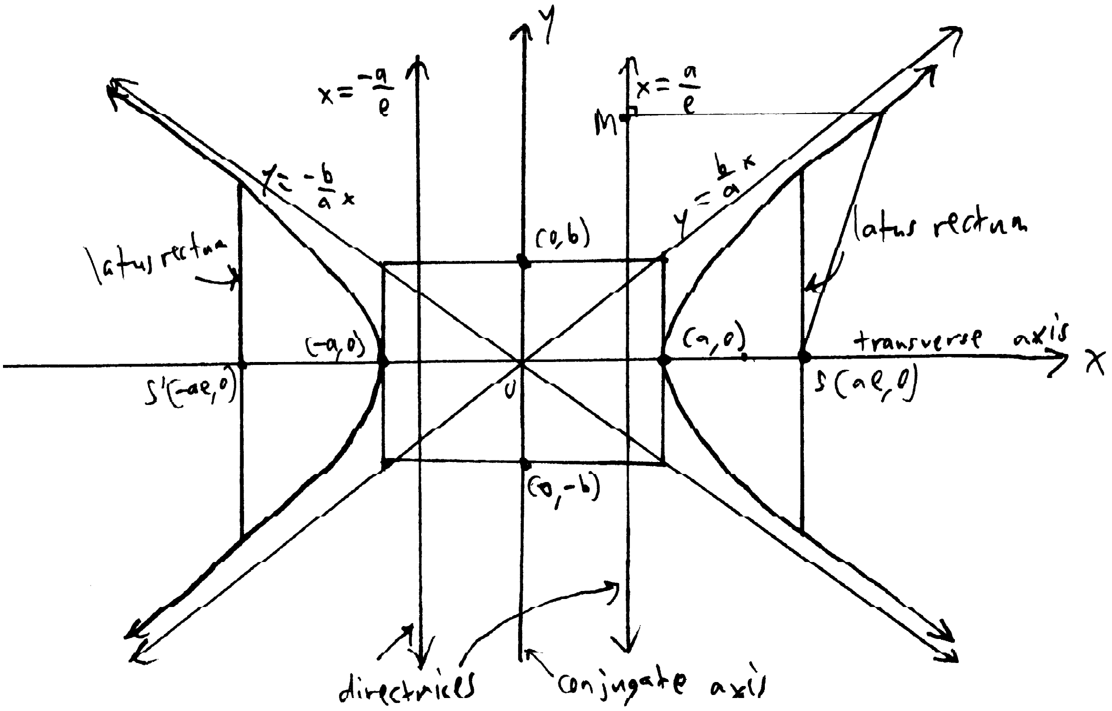
(iii) Equation OQ is $y = \frac{a\sin\theta}{a\cos\theta}x = \frac{\sin\theta}{\cos\theta}x \therefore x\sin\theta - y\cos\theta = 0$ (since O is the origin).
$\text{Norm}_P:\ y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$
and therefore $ax\sin\theta - b^2\cos\theta\sin\theta = ax\sin\theta - a^2\sin\theta\cos\theta$
$\therefore ax\sin\theta - by\cos\theta = (a^2 - b^2)\cos\theta\sin\theta$
$\therefore \frac{ax\sin\theta}{\cos\theta\sin\theta} - \frac{by\cos\theta}{\cos\theta\sin\theta} = \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = ax\sec\theta - by\text{cosec}\,\theta = a^2 - b^2$
$\therefore$ where OQ and Norm_P meet, $x\sin\theta - y\cos\theta = 0$
$\therefore x\sin\theta = y\cos\theta$ and $ax\sin\theta - by\cos\theta = (a^2 - b^2)\cos\theta\sin\theta$
$\therefore ax\sin\theta - bx\sin\theta = (a^2 - b^2)\cos\theta\sin\theta$
$\therefore x\sin\theta(a - b) = (a - b)(a + b)\cos\theta\sin\theta$
$\therefore x\sin\theta = (a + b)\sin\theta\cos\theta$
$\therefore x = (a + b)\cos\theta$ and $y\cos\theta = (a + b)\cos\theta\sin\theta$
$\therefore y = (a+b)\sin\theta$ which are the parametric equations of the circle, centre $(0, 0)$ and radius $(a+b)$ units which has cartesian equation $x^2 + y^2 = (a+b)^2$ and therefore OQ and Norm_P meet on the circle $x^2 + y^2 = (a + b)^2$.

The Hyperbola

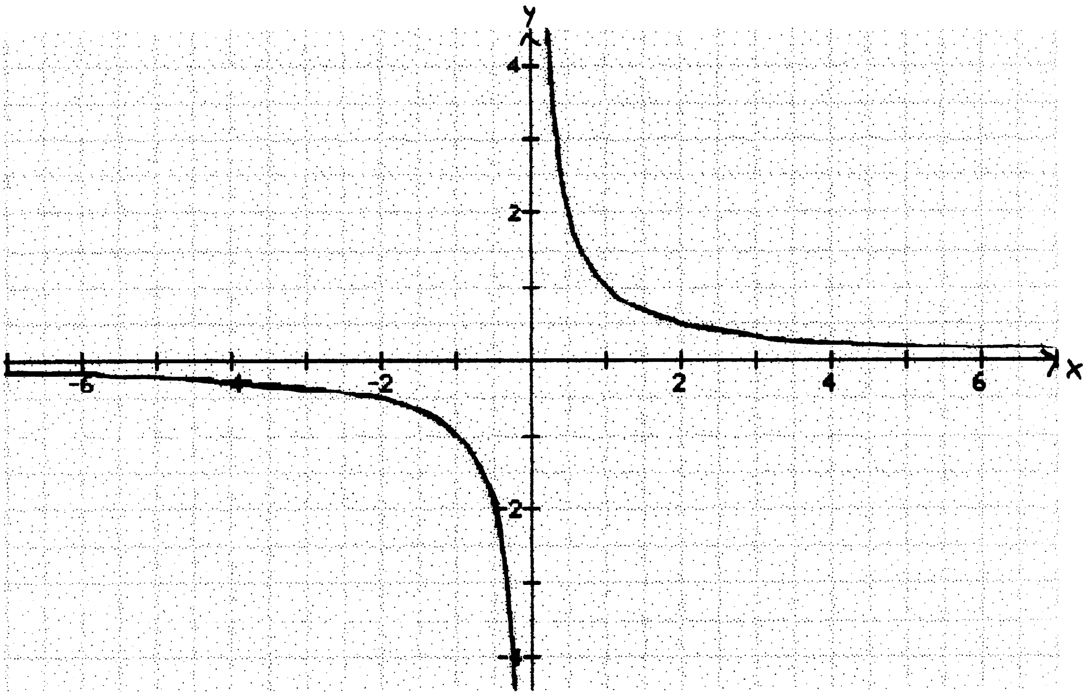
General Equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ eccentricity $e^2 = 1 + \frac{b^2}{a^2} \therefore e = \sqrt{1 + \frac{b^2}{a^2}} > 1$, foci $(\pm ae, 0)$, directrices $x = \pm\frac{a}{e}$, asymptotes $y = \pm\frac{b}{a}x$

Parametric equations

$x = a\sec\theta$, $y = b\tan\theta$ (Note: $e = \frac{|PS|}{|PM|}$ therefore $PS > PM$).

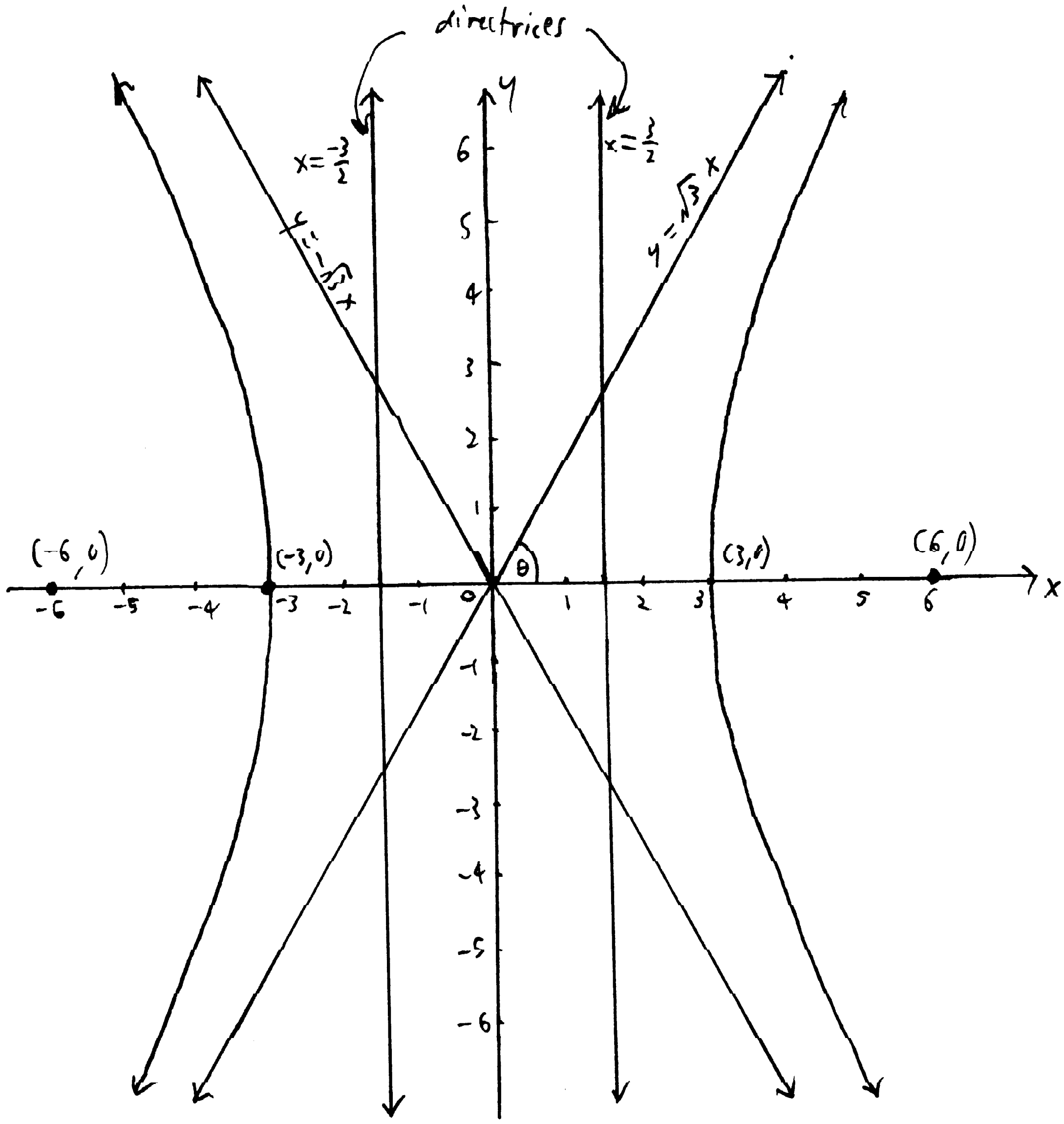


Note $xy = 1$ is a rectangular hyperbola, i.e.,



Example. For the hyperbola $3x^2 - y^2 = 27$ find the vertex, foci, directrices, asymptotes, eccentricity and sketch the hyperbola. Find the acute angle between the asymptotes.

$3x^2 - y^2 = 27.$
$\frac{x^2}{9} - \frac{y^2}{27} = 1$
$a = 3$
$b = \sqrt{27} = 3\sqrt{3}$
Vertices $(3, 0), (-3, 0)$.
$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{27}{9}} = 2$
Foci: $(\pm, 0)$
Directrices: $x = \pm\frac{3}{2}$
Assymptotes: $y = \pm\sqrt{3}x$



$\tan\theta = \sqrt{3}$
$\therefore \theta = 60^\circ$
$\therefore$ $\angle AOB$ is the acute angle between the assymptotes and therefore it is 60°.

Lecture 25

Supp. Set 2F, Q1,9 (Coroneos, 1982b):

1. For the hyperbola $H : x^2/16 - y^2/9 = 1$, find the coordinates of the foci S, S'. P is any point $(4\sec\theta, 3\tan\theta)$ on H.

(i) Show that $PS = 5\sec\theta - 4$ and $PS' = 5\sec\theta + 4$; and hence prove the difference of the focal distances to P is independent of the position of P on H.

(ii) Prove the tangent at P has equation $x\sec\theta/4 - y\tan\theta/3 = 1$. If this tangent meets the transverse axis in T, show that $S'P : PS = S'T : TS$. Deduce that the tangent at P bisects the angle $S'PS$.

(iii) Show the normal at P has equation $4x/\sec\theta + 3y/\tan\theta = 25$. If this normal meets the x-axis in G, prove that $S'P : PS = S'G : SG$.

Solution.

For hyperbola $H : x^2/16 - y^2/9 = 1$, foci S and S are $(\pm\sqrt{16}\sqrt{1+\frac{9}{16}}, 0) = (\pm 5, 0)$ and P on H is $(4\sec\theta, 3\tan\theta)$.

(i) $PS = \sqrt{1+\frac{9}{16}}(4\sec\theta - \frac{\sqrt{16}}{\sqrt{1+\frac{9}{16}}}) = \frac{5}{4}(4\sec\theta - \frac{16}{5}) = 5\sec\theta - 4$
$PS' = \sqrt{1+\frac{9}{16}}(4\sec\theta + \frac{\sqrt{16}}{\sqrt{1+\frac{9}{16}}}) = \frac{5}{4}(4\sec\theta + \frac{16}{5}) = 5\sec\theta + 4$.
$\therefore$ the difference of the focal distances to P is $(5\sec\theta + 4) - (5\sec\theta - 4) = 8$ which is independant of θ and $\therefore$ independant of the position of P on H.

(ii) For H, $\frac{x}{8} - \frac{2y}{9}\frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{9x}{16y} = \frac{36\sec\theta}{48\tan\theta} = \frac{3\sec\theta}{4\tan\theta}$ at P and therefore $\tan_P$ is $y - 3\tan\theta = \frac{3\sec\theta}{4\tan\theta}(x - 4\sec\theta)$ i.e., $4y\tan\theta - 12\tan^2\theta = 3x\sec\theta - 12\sec^2\theta$ i.e., $3x\sec\theta - 4y\tan\theta = 12\sec^2\theta - 12\tan^2\theta = 12(\sec^2\theta - \tan^2\theta) = 12$ i.e., $\frac{x\sec\theta}{4} - \frac{y\tan\theta}{3} = 1$. If this meets transverse axis in T, then for T, $y = 0$ and therefore $\frac{x\sec\theta}{4} = 1$ and therefore $x = \frac{4}{\sec\theta}$ and therefore T is $(\frac{4}{\sec\theta}, 0)$ and threrfore $S'T : TS = (\frac{4}{\sec\theta} + 5) : (5 - \frac{4}{\sec\theta}) = (5\sec\theta + 4) : (5\sec\theta - 4)$. $S'P : PS = (5\sec\theta + 4) : (5\sec\theta - 4) = S'T : TS$ (from **(i)**) and therefore if $S'P$ is produced to I such that $SI||TP$, then $\frac{S'P}{PI} = \frac{S'T}{TS}$ (by intercept theory) $= \frac{S'P}{PS}$ (above) and therefore $\frac{PI}{S'P} = \frac{PS}{S'P}$ and therefore $PI = PS$. Therefore $\triangle IPS$ is isosceles with $\angle PIS = \angle ISP$ (base $\angle$'s of isosceles $\triangle IPS$) $= \angle SPT$ (alternate). But $\angle PIS = \angle S'PT$ (corresponding) and therefore $\angle S'PT = \angle TPS$ and therefore the tangent at P (i.e., TP) bisects $\angle S'PS$.

(iii) Norm_P is $y - 3\tan\theta = \frac{-4\tan\theta}{3\sec\theta}(x - 4\sec\theta)$ and therefore $3y\sec\theta - 9\tan\theta\sec\theta = -4x\tan\theta + 16\sec\theta\tan\theta$ and therefore $4x\tan\theta + 3y\sec\theta = 16\sec\theta\tan\theta + 9\tan\theta\sec\theta = 25\sec\theta\tan\theta$ and therefore $\frac{4x}{\sec\theta} + \frac{3y}{\tan\theta} = 25$. Where G is the x-intercept, $y = 0$ and therefore $\frac{4x}{\sec\theta} = 25$ and therefore $x = \frac{25\sec\theta}{4}$ and therefore G is $(\frac{25\sec\theta}{4}, 0)$ and therefore

$S'G : SG = (\frac{25\sec\theta}{4}+5) : (\frac{25\sec\theta}{4}-5) = (\frac{25\sec\theta+20}{25\sec\theta-20}) = (5\sec\theta+4) : (5\sec\theta-4) = S'P : SP$ and therefore $S'P : SP = S'G : SG$. $\square$

9. Write down the equations of the asymptotes of the hyperbola $H : x^2/9 - y^2/16 = 1$.

(i) Prove that the part of the tangent at $P(3\sec\theta, 4\tan\theta)$ on H which is intercepted between the asymptotes is bisected at P.

(ii) Show that this tangent forms with the asymptotes a triangle of constant area.

(iii) Prove that the tangent intercepted between the asymptotes subtends a constant angle at a focus.

Solution. For the hyperbola $H : x^2/9 - y^2/16 = 1$, asymptotes are $y = \pm\frac{\sqrt{16}}{\sqrt{3}}x = \pm\frac{4}{3}x$.

(i) For $x^2/9-y^2/16 = 1, 2x/9-(2y/16)\frac{dy}{dx} = 0$ and therefore $\frac{dy}{dx} = \frac{16x}{9y} = \frac{16(3\sec\theta)}{9(4\tan\theta)} = \frac{4\sec\theta}{3\tan\theta}$ at $P(3\sec\theta, 4\tan\theta)$ on H and therefore Tan_P is $y-4\tan\theta = \frac{4\sec\theta}{3\tan\theta}(x-3\sec\theta)$ and therefore $3y\tan\theta - 12\tan^2\theta = 4x\sec\theta - 12\sec^2\theta$ and therefore $4x\sec\theta - 3y\tan\theta = 12\sec^2\theta + 2\tan^2\theta = 12(\sec^2\theta - \tan^2\theta) = 12$ and if this tangent crosses the asymptote $y = \frac{4}{3}x$ at A, then for A, $y = \frac{4}{3}x$ and $4x\sec\theta - 3y\tan\theta = 12$ and therefore $4x\sec\theta - 3(\frac{4}{3})\tan\theta = 12$ and therefore $4x\sec\theta - 4x\tan\theta = 12$ and therefore $x(4\sec\theta - 4\tan\theta) = 12$ and therefore $x = \frac{12}{4\sec\theta-4\tan\theta} = \frac{12}{4(\sec\theta-\tan\theta)} = \frac{3}{\sec\theta-\tan\theta}$ and therefore $y = \frac{4}{3}(\frac{3}{\sec\theta-\tan\theta}) = \frac{4}{\sec\theta-\tan\theta}$ and therefore A is $(\frac{3}{\sec\theta-\tan\theta}, \frac{4}{\sec\theta-\tan\theta})$. If Tan_P crosses the asymptote $y = -\frac{4}{3}x$ at B, then for B, $y = -\frac{4}{3}x$ and $4x\sec\theta-3y\tan\theta = 12$ and therefore $4x\sec\theta-3(-\frac{4}{3}x)\tan\theta = 12$ and therefore $4x\sec\theta + 4x\tan\theta = 12$. $\therefore x(4\sec\theta + 4\tan\theta) = 12$. $\therefore x = \frac{12}{4\sec\theta+4\tan\theta} = \frac{12}{4(\sec\theta+\tan\theta)} = \frac{3}{\sec\theta+\tan\theta}$ and therefore $y = -\frac{4}{3}(\frac{3}{\sec\theta+\tan\theta}) = \frac{-4}{\sec\theta+\tan\theta}$ and therefore B is $(\frac{3}{\sec\theta+\tan\theta}, \frac{-4}{\sec\theta+\tan\theta})$ and therefore the midpoint of AB is
$(\frac{\frac{3}{\sec\theta-\tan\theta}+\frac{3}{\sec\theta+\tan\theta}}{2}, \frac{\frac{4}{\sec\theta-\tan\theta}-\frac{4}{\sec\theta+\tan\theta}}{2})$
$= (3(\frac{\frac{1}{\sec\theta-\tan\theta}+\frac{1}{\sec\theta+\tan\theta}}{2}), 4(\frac{\frac{1}{\sec\theta-\tan\theta}-\frac{1}{\sec\theta+\tan\theta}}{2}))$
$= (3(\frac{\sec\theta+\tan\theta+\sec\theta-\tan\theta}{2(\sec^2\theta-\tan^2\theta)}), 4(\frac{\sec\theta+\tan\theta-(\sec\theta-\tan\theta)}{2(\sec^2\theta-\tan^2\theta)}))$
$= (3(\frac{2\sec\theta}{2(1)}), 4(\frac{2\tan\theta}{2(1)})$
$= (3\sec\theta, 4\tan\theta)$
$= P$
and therefore the part of the tangent at $P(3\sec\theta, 4\tan\theta)$ on H which is intercepted between the asymptotes is bisected at P.

(ii) Where O is the origin $(0, 0)$,
$OA = \sqrt{(\frac{3}{\sec\theta-\tan\theta})^2 + (\frac{4}{\sec\theta-\tan\theta})^2}$
$= \sqrt{\frac{9}{(\sec\theta-\tan\theta)^2} + \frac{16}{(\sec\theta-\tan\theta)^2}}$
$= \sqrt{\frac{25}{(\sec\theta-\tan\theta)^2}}$
$= \frac{5}{\sec\theta-\tan\theta}$.

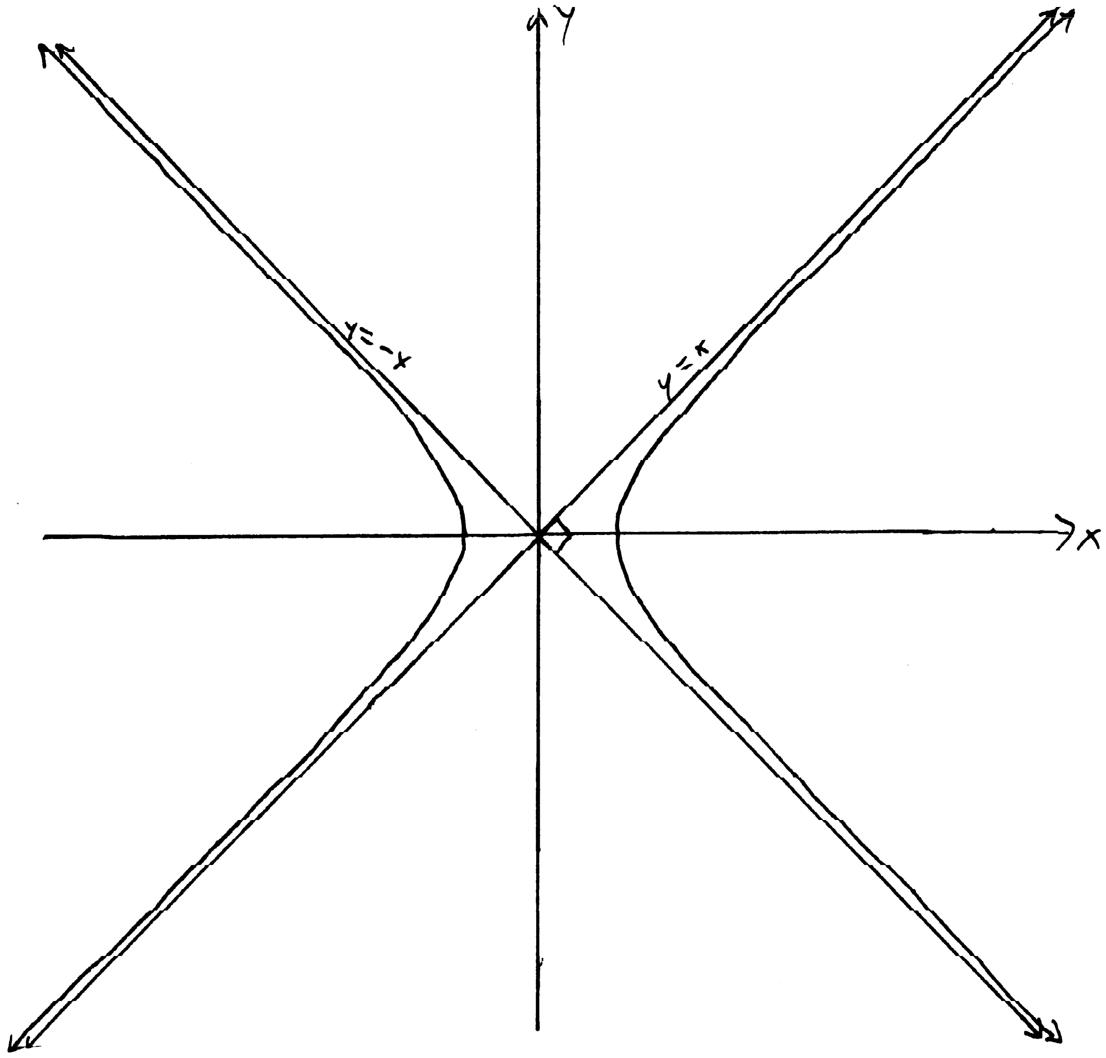
$OB = \sqrt{(\frac{3}{\sec\theta+\tan\theta})^2 + (\frac{4}{\sec\theta+\tan\theta})^2}$
$= \sqrt{\frac{9}{(\sec\theta+\tan\theta)^2} + \frac{16}{(\sec\theta+\tan\theta)^2}}$
$= \sqrt{\frac{25}{(\sec\theta+\tan\theta)^2}}$
$= \frac{5}{\sec\theta+\tan\theta}$.
$\angle AOB = 2\angle AOX$
$= 2\tan^{-1}(\frac{4}{\sec\theta-\tan\theta}/\frac{3}{\sec\theta-\tan\theta})$
$= 2\tan^{-1}\frac{4}{3}$
$\therefore$ Area $\triangle AOB = \frac{1}{2}\cdot OB\cdot OB\cdot\sin\angle AOB$
$= \frac{1}{2}\cdot\frac{5}{\sec\theta-\tan\theta}\cdot\frac{5}{\sec\theta+\tan\theta}\cdot\sin(2\tan^{-1}\frac{4}{3})$
$= \frac{25}{\sec^2\theta-\tan^2\theta}(\sin\tan^{-1}\frac{4}{3}\cos\tan^{-1}\frac{4}{3})$
$= 25(\frac{4}{\sqrt{4^2+3^2}}\cdot\frac{3}{\sqrt{4^2+3^2}})\text{unit}^2$
$= 25(\frac{12}{4^2+3^2})\text{unit}^2$
$= 25(\frac{12}{25}\text{unit}^2$
$= 12\text{unit}^2$ which is a constant and therefore the tangent forms with the asymptotes of a triangle of constant area.

(iii) The focus S is $(\sqrt{9}, \sqrt{1+\frac{16}{9}}, 0) = (3(\frac{5}{3}), 0) = (5, 0)$ and therefore
$\angle ASB = 180° - \tan^{-1}\frac{((\frac{4}{\sec\theta-\tan\theta}-0)/(\frac{3}{\sec\theta-\tan\theta}-5))-((\frac{-4}{\sec\theta+\tan\theta}-0)/(\frac{3}{\sec\theta+\tan\theta}-5))}{1+((\frac{4}{\sec\theta-\tan\theta}-0)/(\frac{3}{\sec\theta-\tan\theta}-5))((\frac{-4}{\sec\theta+\tan\theta}-0)/(\frac{3}{\sec\theta+\tan\theta}-5))}$
$= 180° - \tan^{-1}\frac{((\frac{4}{\sec\theta-\tan\theta})/(\frac{3-5(\sec\theta-\tan\theta)}{\sec\theta-\tan\theta}))-((\frac{-4}{\sec\theta+\tan\theta})/(\frac{3-4(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}))}{1+((\frac{4}{\sec\theta-\tan\theta})/(\frac{3-5(\sec\theta-\tan\theta)}{\sec\theta-\tan\theta}))((\frac{-4}{\sec\theta+\tan\theta})/(\frac{3-5(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}))}$
$= 180° - \tan^{-1}\frac{(4/(3-5(\sec\theta-\tan\theta)))+(4/(3-5(\sec\theta+\tan\theta)))}{1-(4/(3-5(\sec\theta-\tan\theta)))(4/(3-5(\sec\theta+\tan\theta)))}$
$= 180° - \tan^{-1}\frac{(4(3-5(\sec\theta+\tan\theta))+4(3-5(\sec\theta-\tan\theta)))/((3-5(\sec\theta-\tan\theta)(3-5(\sec\theta+\tan\theta)))}{((3-5(\sec\theta-\tan\theta))(3-5(\sec\theta+\tan\theta))-4(4))/((3-5(\sec\theta-\tan\theta)(3-5(\sec\theta+\tan\theta)))}$
$= 180° - \tan^{-1}\frac{(4(3-5(\sec\theta+\tan\theta))+4(3-5(\sec\theta-\tan\theta)))}{((3-5(\sec\theta-\tan\theta))(3-5(\sec\theta+\tan\theta))-4(4)}$
$= 180° - \tan^{-1}\frac{12-20\sec\theta-20\tan\theta+12-20\sec\theta+20\tan\theta}{9-15(\sec\theta+\tan\theta)-15(\sec\theta-\tan\theta)+25(\sec^2\theta-\tan^2\theta)-16}$
$= 180° - \tan^{-1}\frac{24-20\sec\theta}{25(1)-7-15\sec\theta-15\tan\theta-15\sec\theta+15\tan\theta}$
$= 180° - \tan^{-1}\frac{4(6-10\sec\theta)}{18-30\sec\theta}$
$= 180° - \tan^{-1}\frac{4(6-10\sec\theta)}{3(6-10\sec\theta)}$
$= 180° - \tan^{-1}\frac{4}{3}$
$= 180° - 53°8'$
$= 126°52'$ which is a constant and therefore the tangent intercepted between the asymptotes subtends a constant angle at a focus ($53°8'$ at one focus and $126°52'$ at the other).

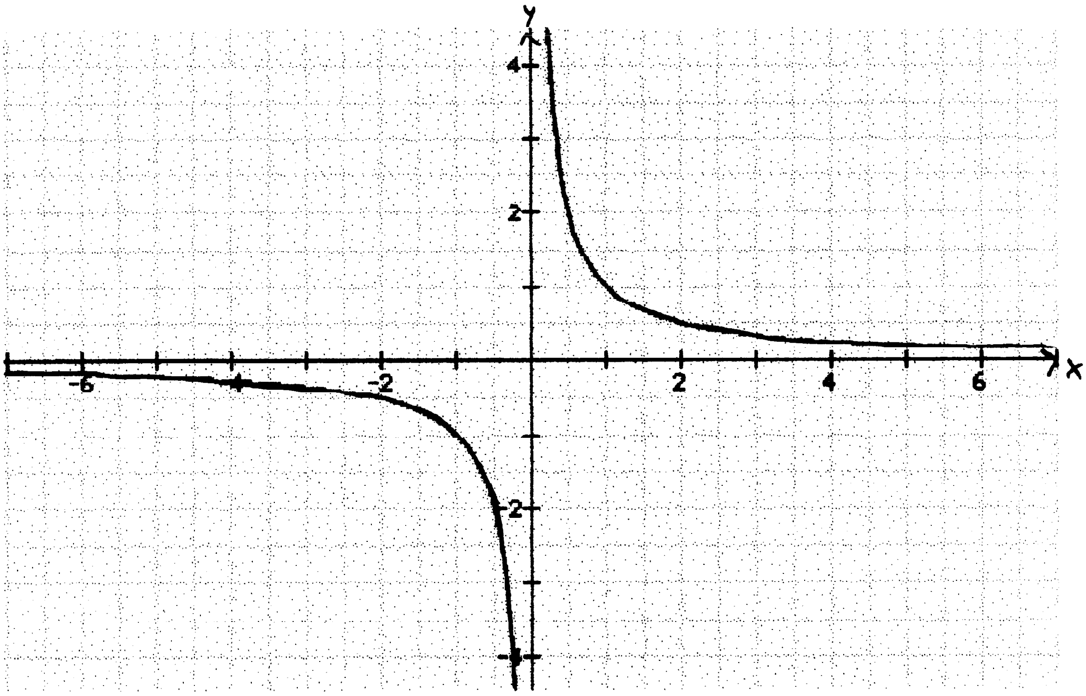
Lecture 26

Rectangualar Hyperbola - asymptotes are at 90° to each other.

$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$ or $x^2 - y^2 = a^2$ i.e., asymptotes: $y = \pm x$, eccentricity: $e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{2}$.
For



swiveled around the origin, we get $xy = c^2$ of parametric equations $x = cp, y = \frac{c}{p}$



Tangents and normals at $P(cp, \frac{c}{p})$:
$xy = c^2$
$y = c^2x^{-1}$
$\therefore \frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2} = -\frac{c^2}{(cp)^2}(a + p) = -\frac{1}{p^2}.$
Tan_P: $y - \frac{c}{p} = -\frac{1}{p}(x - cp)$
$x + p^2y = cp + cp$
$\therefore x + p^2y = 2cp$
Norm_P: $y - \frac{c}{p} = p^2(x - cp)$
$p^3x - py = p^4c - c = c(p^4 - 1).$